# Lösungen für 16.Übung Mathematik Sommersemester

**Aufgabe 1:**

a) lineare Substitution ===> $F(x) = -\frac{1}{\omega}\cos(\omega t) + c$

b) lineare Substitution ===> $F(x) = -\frac{3}{\omega}\cos(\omega t + \varphi) + c$

c) lineare Substitution ===> $F(x) = \frac{(a+bx)^{n+1}}{b \cdot (n+1)} + c$

d) lineare Substitution ===> $F(x) = \frac{1}{a}e^{ax} + c$

e) Substitution für Funktion des Typs $\frac{f'(x)}{f(x)}$: $f(x) = 1 + x^3$ ; $f'(x) = 3x^2$

$\Rightarrow \quad F(x) = \frac{1}{3}\ln\left|1 + x^3\right| + c$

f) Substitution $u = a^x \Rightarrow \ln u = x \cdot \ln a \Rightarrow x = \frac{\ln u}{\ln a} = g(u) \quad \Rightarrow \quad g'(u) = \frac{1}{u \cdot \ln a}$

$$\Rightarrow \quad \int \frac{u \cdot \sqrt{1+u}}{u \cdot \ln a}\,du = \frac{1}{\ln a} \cdot \int \sqrt{1+u}\,du = \frac{1}{\ln a} \cdot \frac{2}{3} \cdot (1+u)^{\frac{3}{2}} + c = \frac{2}{3 \cdot \ln a}\left(1 + a^x\right)^{\frac{3}{2}} + c$$

g) Substitution: $u = -x^3 \Rightarrow x = -\sqrt[3]{u} = -u^{\frac{1}{3}} = g(u) \quad \Rightarrow \quad g'(u) = -\frac{1}{3} \cdot u^{-\frac{2}{3}}$

$$\Rightarrow \quad \int u^{\frac{2}{3}} \cdot e^u \cdot \left(-\frac{1}{3}\right) \cdot u^{-\frac{2}{3}}\,du = -\frac{1}{3} \cdot \int e^u\,du = -\frac{1}{3} \cdot e^{-x^3} + c$$

**Aufgabe 2:**

a) lineare Substitution: $u = bx \quad \Rightarrow \quad \int a^{bx}\,dx = \frac{1}{b} \cdot \frac{a^{bx}}{\ln a} + c$

b) lineare Substitution: $u = \omega t - \varphi \quad \Rightarrow$

$$\int \cos^2(\omega t - \varphi)\,dt = \frac{1}{\omega} \cdot \int \cos^2 u\,du = \frac{1}{\omega} \cdot \left(\frac{1}{2} \cdot \cos u \cdot \sin u + \frac{1}{2}u\right) + c$$

$$= \frac{1}{2\omega} \cdot \left(\omega t - \varphi + \cos(\omega t - \varphi) \cdot \sin(\omega t - \varphi)\right) + c$$

$$= \frac{1}{2\omega} \cdot \left(\omega t - \varphi + \frac{1}{2} \cdot \sin 2(\omega t - \varphi)\right) + c$$

c) lineare Substitution: $u = \omega t - \varphi \quad \Rightarrow$ analog zu b) nur ist jetzt ω die Integrationsvariable!

$$\int \cos^2(\omega t - \varphi)\,d\omega = \frac{1}{2t} \cdot \left(\omega t - \varphi + \frac{1}{2} \cdot \sin 2(\omega t - \varphi)\right) + c$$

d) lineare Substitution: $u = \omega t - \varphi \quad \Rightarrow$ analog zu b) nur ist jetzt φ die Integrationsvariable!

$$\int \cos^2(\omega t - \varphi)\, d\varphi = -\frac{1}{2}\cdot\left(\omega t - \varphi + \frac{1}{2}\cdot\sin 2(\omega t - \varphi)\right) + c$$

e) Substitution $\quad u = 3p^2 - 7 \Rightarrow p = \sqrt{\frac{u}{3}+\frac{7}{3}} \quad \Rightarrow \quad dp = \frac{1}{6p}du$

$$\int p\cdot\left(3p^2-7\right)^3 dp = \frac{1}{6}\cdot\int\frac{p\cdot u^3}{p}du = \frac{1}{6}\cdot\int u^3\, du = \frac{1}{24}u^4 + c = \frac{1}{24}\cdot\left(3p^2-7\right)^4 + c$$

**Aufgabe 3:** a) $\displaystyle\int_0^{2\pi}(1+\sin z)\,dz = [z - \cos z]_0^{2\pi} = 2\pi$

b) $\displaystyle\int_0^{2\pi}(u+\sin u)\,du = \left[\frac{1}{2}u^2 - \cos u\right]_0^{2\pi} = 2\pi^2$

c) $\displaystyle\int_{-a}^{a} a^x\, dx = \left[\frac{a^x}{\ln a}\right]_{-a}^{a} = \frac{a^a - a^{-a}}{\ln a}$

d) Substitution vom Typ $[f(x)]^n \cdot f'(x)$ mit $f(x) = \sin x$ , $n = 5$ , $f'(x) = \cos x$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\sin^5 x\cdot coxdx = \frac{1}{6}\left[\sin^6 x\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} = \frac{1}{6}\cdot\left(\sin^6\frac{\pi}{2} - \sin^6\frac{\pi}{4}\right) = \frac{1}{6}\cdot\left(1 - \left(\frac{\sqrt{2}}{2}\right)^6\right)$$

$$= \frac{1}{6}\cdot\left(1 - \frac{2^3}{2^6}\right) = \frac{1}{6}\cdot\left(1 - \frac{1}{8}\right) = \frac{1}{6}\cdot\frac{7}{8} = \frac{7}{48}$$

e) Substitution $\quad x = \sin u \quad \Rightarrow \quad dx = \cos u\, du$

Substitution der Grenzen: $\quad 0 \to \sin u = 0 \to u = 0 \quad und \quad 1 \to \sin u = 1 \to u = \frac{\pi}{2}$

$$\int_0^1 x^2\cdot\sqrt{1-x^2}\, dx = \int_0^{\frac{\pi}{2}}\sin^2 u\cdot\sqrt{1-\sin^2 u}\cdot\cos u\, du = \int_0^{\frac{\pi}{2}}\sin^2 u\cdot\cos^2 u\, du = \frac{1}{4}\int_0^{\frac{\pi}{2}}\sin^2 2u\, du$$

**Beziehung:** $\quad \sin u\cdot\cos u = \frac{1}{2}\cdot\sin 2u$

erneute Substitution, diesmal linear: v=2u ; Grenzen: $0 \to 0$ und $\frac{\pi}{2} \to \pi$

$$\frac{1}{4}\int_0^{\frac{\pi}{2}}\sin^2 2u\, du = \frac{1}{8}\int_0^{\pi}\sin^2 v\, dv = \frac{1}{16}\int_0^{\pi}(1-\cos 2v)\, dv$$

**Beziehung:** $\quad \sin^2 v = \frac{1}{2}\cdot(1-\cos 2v)$

erneute lineare Substitution: t=2v ; Grenzen: $0 \to 0$ und $\pi \to 2\pi$

$$\frac{1}{16}\int_0^{\pi}(1-\cos 2v)\,dv = \frac{1}{32}\int_0^{2\pi}\left(1-\cos t\right)dt = \frac{1}{32}\left[t-\sin t\right]_0^{2\pi} = \frac{1}{32}\cdot\left(2\pi - \sin 2\pi + \sin 0\right) = \frac{\pi}{16}$$

**Aufgabe 4:**

**a)** partielle Integration: $p' = e^v \quad p = e^v$; $q = \cos v \quad q' = -\sin v$

$$\int e^v \cdot \cos v\,dv = e^v \cdot \cos v + \int e^v \cdot \sin v\,dv$$

$p' = e^v \quad p = e^v$; $q = \sin v \quad q' = \cos v$

$$\int e^v \cdot \sin v\,dv = e^v \cdot \sin v - \int e^v \cdot \cos v\,dv$$

$$\Rightarrow \quad 2\int e^v \cdot \cos v\,dv = e^v \cdot \cos v + e^v \cdot \sin v + c$$

$$\Rightarrow \quad \int e^v \cdot \cos v\,dv = \frac{e^v}{2}\left(\cos v + \sin v\right) + c$$

**b)** Substitution:

$$u = \sqrt{1-\sin x} \Rightarrow u^2 = 1-\sin x \Rightarrow \sin x = 1-u^2 \Rightarrow \cos x = \sqrt{1-\sin^2 x} = \sqrt{1-\left(1-u^2\right)^2}$$

$$x = \arcsin\left(1-u^2\right) \quad \Rightarrow \quad dx = \frac{-2u}{\sqrt{1-\left(1-u^2\right)^2}}du$$

$$F(x) = \int \frac{\sqrt{1-\left(1-u^2\right)^2}}{u}\cdot\frac{-2u}{\sqrt{1-\left(1-u^2\right)^2}}du = -2\int du = -2u + c = -2\cdot\sqrt{1-\sin x} + c$$

**c)** Substitution: $v = \sin u \quad \Rightarrow \quad u = \arcsin v \quad \Rightarrow \quad du = \frac{1}{\sqrt{1-v^2}}dv$

außerdem gilt: $\cos u = \sqrt{1-\sin^2 u} = \sqrt{1-v^2}$

$$\int \cos u \cdot \sin u \cdot e^{\sin^2 u}\,du = \int \sqrt{1-v^2}\cdot v \cdot e^{v^2}\,\frac{1}{\sqrt{1-v^2}}dv = \int v \cdot e^{v^2}\,dv$$

Substitution: $t = v^2 \quad \Rightarrow \quad v = \sqrt{t} \quad \Rightarrow \quad dv = \frac{1}{2\cdot\sqrt{t}}dt$

$$\int v \cdot e^{v^2}\,dv = \frac{1}{2}\int \sqrt{t}\cdot e^t \cdot \frac{1}{\sqrt{t}}dt = \frac{1}{2}e^t + c = \frac{1}{2}e^{v^2} + c = \frac{1}{2}e^{\sin^2 u} + c$$

**Aufgabe 5:**

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx = -\int_0^{-a} f(x)\,dx + \int_0^a f(x)\,dx = \int_0^a f(-u)\,du + \int_0^a f(x)\,dx$$

lin.Substitution u = -x bzw. x = -u für das 1.Teilintegral

dx = -du

Substitution der Grenzen: $0 \to 0$ und $-a \to a$

f(x) ungerade $\Rightarrow$ f(-u) = -f(u)

$$\Rightarrow \quad -\int_0^a f(u)\,du + \int_0^a f(x)\,dx = -\int_0^a f(x)\,dx + \int_0^a f(x)\,dx = 0$$

f(x) gerade $\Rightarrow$ f(-u) = f(u)

$$\Rightarrow \quad \int_0^a f(u)\,du + \int_0^a f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(x)\,dx = 2\cdot\int_0^a f(x)\,dx$$

**Aufgabe 6:** a) lineare Substitution: $u = 4x - 2 \quad ; \quad dx = \frac{1}{4}du$

$$\int f(x)\,dx = \frac{3}{4}\int \frac{du}{\cos^2 u} = \frac{3}{4}\cdot \tan u + c = \frac{3}{4}\cdot\tan(4x-2) + c$$

b) partielle Integration:

$$u = e^{ax} \qquad u' = a\cdot e^{ax}$$

$$v' = \sin bx \qquad v = -\frac{1}{b}\cdot\cos bx$$

$$\int e^{ax}\cdot\sin bx\,dx = -\frac{1}{b}\cdot e^{ax}\cdot\cos bx + \frac{a}{b}\int e^{ax}\cdot\cos bx\,dx$$

$$u = e^{ax} \qquad u' = a\cdot e^{ax}$$

$$v' = \cos bx \qquad v = \frac{1}{b}\cdot\sin bx$$

$$\frac{a}{b}\cdot\int e^{ax}\cdot\cos bx\,dx = \frac{a}{b^2}\cdot e^{ax}\cdot\sin bx - \frac{a^2}{b^2}\cdot\int e^{ax}\cdot\sin bx\,dx$$

$$\int e^{ax}\cdot\sin bx\,dx = \frac{e^{ax}}{b}\cdot\left(\frac{a}{b}\cdot\sin bx - \cos bx\right) - \frac{a^2}{b^2}\cdot\int e^{ax}\cdot\sin bx\,dx + c$$

$$\frac{a^2+b^2}{b^2}\cdot\int e^{ax}\cdot\sin bx\,dx = \frac{e^{ax}}{b}\cdot\left(\frac{a}{b}\cdot\sin bx - \cos bx\right) + c$$

$$\int e^{ax}\cdot\sin bx\,dx = \frac{b\cdot e^{ax}}{a^2+b^2}\cdot\left(\frac{a}{b}\cdot\sin bx - \cos bx\right) + c$$

$$= \frac{e^{ax}}{a^2+b^2}\cdot(a\cdot\sin bx - b\cdot\cos bx) + c$$

**Aufgabe 7:** lineare Substitution: $u = 4 - 3x \quad ; \quad dx = -\frac{1}{3}du$

Substitution der Grenzen: $x_1 = -2 \rightarrow u_1 = 10$; $x_2 = -\frac{1}{3} \rightarrow u_2 = 5$

$$\int_{-2}^{-\frac{1}{3}} f(x)\,dx = -\frac{2}{3}\cdot\int_{10}^{5}\frac{du}{u} = -\frac{2}{3}\cdot\left[\ln|u|\right]_{10}^{5} = -\frac{2}{3}\cdot(\ln 5 - \ln 10) = \frac{2}{3}\cdot\ln\frac{10}{5} = \frac{2}{3}\cdot\ln 2 = 0{,}462$$